➔ Regular Research Paper – NS

Integral representations for the solutions of the quadratic pencil of the Sturm-Liouville equation with a discontinuous coefficient

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Abstract

In this study we construct new integral representations of Jost-type solutions of the quadratic pencil of the Sturm-Liouville equation with piece-wise constans coefficient on the entire axis under some boundness conditions on the potential functions.

Keywords: *Jost-type solution, Sturm-Liouville pencil, discontinious Sturm-Liouville equation, integral equation, transformation operator.*

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1. INTRODUCTION

In the present study the Sturm-Liouville equation

$$-y'' + q(x)y + 2\lambda p(x)y = \lambda^2 \rho(x)y, x \in I = (-\infty, +\infty) \tag{1}$$

is considered where

$$\rho(x) = \begin{cases} 1, & x \geq 0, \\ \alpha^2, & x < 0, \end{cases} (\alpha \neq 1, \alpha > 0) \tag{2}$$

is a piece-wise constant coefficient, λ is a complex parameter, $q(x)$ and $p(x)$ are real functions such that

$$(1 + |x|)q(x), p(x) \in L^1(I), p(x) \in BC(I) \tag{3}$$

Here $L^1(I)$ is the space of summable functions on I and $BC(I)$ is the class of functions that are bounded and continuous on I . Equation (1) is related to solving the inverse problem for the Klein-Gordon equation with a static potential and zero charge in quantum scattering theory [7]. Some scattering problems arising in the theory of transmission lines, the theory of electromagnetism, and the theory of elasticity are also reduced to equation (1) [12]. It is well known that transformation operators method plays an important role in the inverse problems theory. V.A. Marchenko [1, 8] applied the transformation operators to the solution of the inverse problems for Sturm-Liouviile operator on a finite interval and on the half line. Transformation operators were also used in the study of Levitan, Gasymov [1], where they obtained necessary and sufficient conditions for recovering a Sturm-Liouville operator from its spectral characteristics. In the case of $\rho(x)=1$, there are enough studies in the literature using transformation type operators, called integral representations of the special solutions, to solve direct and inverse scattering problems of equation (1) [3, 6, 7, 11] . Some problems with various statements related to inverse scattering problems for the discontinuous Sturm-Liouville equation have been considered in [2, 4, 5, 9]. The direct and inverse scattering problems for equation (1) with $p(x)=0$ in various settings have been investigated in [4, 5, 13] where new integral representations, similar transformations operators for the Jost solutions of the Sturm-Liouville equation, are obtained and applied to the investigation of the considered problems. In [10] direct and inverse scattering problems have been investigated for equation (1) with discontinuity conditions. In this study we construct new integral representations of Jost-type solutions of equation (1) on the entire axis under conditions (2) and (3).

2. INTEGRAL REPRESENTATION OF THE JOST SOLUTIONS

We denote by $f_{\pm}(x,\lambda)$ the solution of (1) with the condition

$$\lim_{x\to\pm\infty} f_{\pm}(x,\lambda)\, e^{\mp\lambda\mu(x)} = 1,$$

where $\mu(x)=x\sqrt{\rho(x)}$. The solutions $f_{+}(x,\lambda)$ and $f_{-}(x,\lambda)$ will be called the right and the left Jost solutions of (1) respectively. It is easy to verify that the solution $f_{\pm}(x,\lambda)$ obeys the integral equation

$$f_{\pm}(x,\lambda)=e_{\pm}(x,\lambda)+\int_{x}^{\pm\infty}\left[\frac{1}{2}\left(\frac{1}{\sqrt{\rho(t)}}-\frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda(\mu(t)+\mu(x))}{\lambda}+\right. \quad (4)$$

$$\left.+\frac{1}{2}\left(\frac{1}{\sqrt{\rho(t)}}+\frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda(\mu(t)-\mu(x))}{\lambda}\right]\mathrm{x}(q(t)+2\lambda p(t)f_{\pm}(t,\lambda))dt,$$

where

$$e_{+}(x,\lambda)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu(x)}+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)e^{-i\lambda\mu(x)}$$

and

$$e_{-}(x,\lambda)=\frac{1}{2}\left(1-\frac{\alpha}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu(x)}+\frac{1}{2}\left(1+\frac{\alpha}{\sqrt{\rho(x)}}\right)e^{-i\lambda\mu(x)}$$

Consider the solution $f_{+}(x,\lambda)$ When $x>0$ it is well known that [3, 6] for all $Im\lambda\geq 0$ the solution $f_{+}(x,\lambda)$ has the representation

$$f_{+}(x,\lambda)=e^{i\lambda x+i\omega_{+}(x)}+\int_{x}^{+\infty}A^{+}(x,t)e^{i\lambda t}dt \quad (5)$$

where $\omega_{+}(x)=\int_{x}^{+\infty}p(t)dt$ and the kernel function $A^{+}(x,t)$ satisfies

$$\int_{x}^{+\infty}|A^{+}(x,t)|\leq C_0\sigma^{+}(x)e^{\sigma^{+}(x)} \quad (6)$$

for some constant $C_0>0$ and $\sigma^{+}(x)=\int_{x}^{+\infty}\left((1+t)|q(t)|+2|p(t)|\right)$. Moreover, the kernel function $A^{+}(x,t)$ satisfies the condition

$$A^{+}(x,x)=\frac{1}{2}\left(\int_{x}^{+\infty}[q(t)+p^{2}(t)]dt-ip(x)\right)e^{i\omega_{+}(x)} \quad (7)$$

Consider the case $x<0$ for the solution $f_{+}(x,\lambda)$ In this case the equation (4) takes the form of

$$f_+(x,\lambda) = \alpha_+ e^{i\alpha\lambda x} + \alpha_- e^{-i\alpha\lambda x} + \int_x^0 \frac{\sin\alpha\lambda(t-x)}{\alpha\lambda} q(t) f_+(t,\lambda)dt +$$
$$+\int_0^\infty \left[\alpha^- \frac{\sin\lambda(t+\alpha x)}{\lambda} + \alpha^+ \frac{\sin\lambda(t-\alpha x)}{\lambda}\right][q(t)+2\lambda p(t)] f_+(t,\lambda)dt, \quad (8)$$

where $\alpha_\pm = \frac{1}{2}(1 \pm \frac{1}{\alpha})$. We require that the solution of the integral equation (8) has the form of

$$f_+(x,\lambda) = R_+(x)e^{i\alpha\lambda x} + R_-(x)e^{-i\alpha\lambda x} + \int_{\alpha x}^{+\infty} B^+(x,t)e^{i\lambda t}dt, \qquad Im\lambda \geq 0, x < 0 \quad (9)$$

where

$$R_\pm(x) = \alpha_\pm e^{i\omega_+(0) \pm \frac{i}{\alpha}\int_x^0 p(t)dt}$$

and $B^+(x,t)$ is defined after replacing $f_+(x,\lambda)$ in equation (8) with formulas (5), (9) and transforming some integrals of the Fourier type:

$$B^+(x,t) = \frac{1}{2\alpha}\int_{\frac{t+\alpha x}{2\alpha}}^0 q(s)R_+(s)ds + \frac{1}{2\alpha}\int_{\frac{\alpha x - t}{2\alpha}}^0 q(s)R_-(s)ds -$$
$$-\frac{i}{2\alpha^2} p\left(\frac{t+\alpha x}{2\alpha}\right) R_+\left(\frac{t+\alpha x}{2\alpha}\right) + \frac{i}{2\alpha^2} p\left(\frac{\alpha x - t}{2\alpha}\right) R_-\left(\frac{\alpha x - t}{2\alpha}\right) +$$
$$+\frac{\alpha_+}{2}\int_0^{+\infty} q(s)\, e^{i\omega_+(s)}ds - \frac{\alpha_-}{2}\int_0^{\frac{t-\alpha x}{2}} q(s)\, e^{i\omega_+(s)}ds - \frac{i\alpha_-}{2} p\left(\frac{t-\alpha x}{2}\right) e^{i\omega_+\left(\frac{t-\alpha x}{2}\right)} +$$
$$+\alpha_+ \int_0^{\frac{t-\alpha x}{2}} dv \int_v^{+\infty} q(u-v)A^+(u-v,u+v)du -$$
$$-\alpha_- \int_0^{\frac{t-\alpha x}{2}} dv \int_v^{\frac{t-\alpha x}{2}} q(u-v)A^+(u-v,u+v)du +$$
$$+\frac{1}{\alpha^2}\int_0^{\frac{t-\alpha x}{2}} dv \int_{\frac{t+\alpha x}{2}}^{v} q\left(\frac{u-v}{\alpha}\right) B^+\left(\frac{u-v}{\alpha}, u+v\right) du +$$
$$+i\alpha_+ \int_{\frac{t-\alpha x}{2}}^{+\infty} p\left(u - \frac{t-\alpha x}{2}\right) A^+\left(u - \frac{t-\alpha x}{2}, u + \frac{t-\alpha x}{2}\right) du -$$
$$-i\alpha_- \int_0^{\frac{t-\alpha x}{2}} p\left(\frac{t-\alpha x}{2} - v\right) A^+\left(\frac{t-\alpha x}{2} - v, \frac{t-\alpha x}{2} + v\right) dv +$$

$$+\frac{1}{i\alpha^2}\int_0^{\frac{t-\alpha x}{2}} p\left(\frac{t+\alpha x}{2}-\frac{v}{\alpha}\right)B^+\left(\frac{t+\alpha x}{2\alpha}-\frac{v}{\alpha},\frac{t+\alpha x}{2}+v\right)dv-$$

$$-\frac{1}{i\alpha^2}\int_0^{\frac{t-\alpha x}{2}} p\left(\frac{u}{\alpha}-\frac{t-\alpha x}{2\alpha}\right)B^+\left(\frac{u}{\alpha}-\frac{t-\alpha x}{2\alpha},\frac{u}{\alpha}+\frac{t+\alpha x}{2}\right)du, -\alpha x\le t<-\alpha x \quad (10)$$

$$B^+(x,t)=\frac{\alpha_+}{2}\int_{\frac{t+\alpha x}{2}}^{+\infty} q(s)ds+\frac{\alpha_-}{2}\int_{\frac{t-\alpha x}{2}}^{+\infty} q(s)ds-$$

$$-\frac{i\alpha_-}{2}p\left(\frac{t-\alpha x}{2}\right)e^{i\omega_+\left(\frac{t-\alpha x}{2}\right)}-\frac{i\alpha_+}{2}p\left(\frac{t+\alpha x}{2}\right)e^{i\omega_+\left(\frac{t+\alpha x}{2}\right)}$$

$$\alpha_+\int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} dv\int_v^{+\infty} q(u-v)A^+(u-v,u+v)du+$$

$$\alpha_+\int_0^{\frac{t-\alpha x}{2}} dv\int_{\frac{t+\alpha x}{2}}^{+\infty} q(u-v)A^+(u-v,u+v)du-$$

$$-\alpha_-\int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} dv\int_v^{\frac{t-\alpha x}{2}} q(u-v)A^+(u-v,u+v)du+$$

$$+\frac{1}{\alpha^2}\int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} dv\int_{\frac{t+\alpha x}{2}}^{v} q\left(\frac{u-v}{\alpha}\right)B^+\left(\frac{u-v}{\alpha},u+v\right)dv+$$

$$+i\alpha_+\int_{\frac{t-\alpha x}{2}}^{+\infty} p\left(u-\frac{t-\alpha x}{2}\right)A^+\left(u-\frac{t-\alpha x}{2},u+\frac{t-\alpha x}{2}\right)du-$$

$$-i\alpha_+\int_0^{\frac{t+\alpha x}{2}} p\left(\frac{t+\alpha x}{2}-v\right)A^+\left(\frac{t+\alpha x}{2}-v,\frac{t+\alpha x}{2}+v\right)dv+$$

$$-i\alpha_-\int_0^{\frac{t-\alpha x}{2}} p\left(\frac{t-\alpha x}{2}-v\right)A^+\left(\frac{t-\alpha x}{2}-v,\frac{t-\alpha x}{2}+v\right)dv+$$

$$+i\alpha_{-}\int_{\frac{t+\alpha x}{2}}^{+\infty} p\left(u-\frac{t+\alpha x}{2}\right)A^{+}\left(u-\frac{t+\alpha x}{2},u+\frac{t+\alpha x}{2}\right)du+$$

$$+\frac{1}{i\alpha^2}\int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} p\left(\frac{t+\alpha x}{2\alpha}-\frac{v}{\alpha}\right)B^{+}\left(\frac{t+\alpha x}{2\alpha}-\frac{v}{\alpha},\frac{t+\alpha x}{2}+v\right)dv-$$

$$-\frac{1}{i\alpha^2}\int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} p\left(\frac{u}{\alpha}-\frac{t-\alpha x}{2\alpha}\right)B^{+}\left(\frac{u}{\alpha}-\frac{t-\alpha x}{2\alpha},u+\frac{t-\alpha x}{2}\right)du, t>-\alpha x. \quad (11)$$

Here we suppose $A^{+}(x,t)\equiv 0$ for $t<0$ and $B^{+}(x,t)\equiv 0$ for $t<\alpha x$. From equation (11) it is obtained that

$$\int_{\alpha x}^{+\infty}|B^{+}(x,t)|\,dt\le C\sigma^{+}(x)e^{\sigma^{+}(x)}$$

where $C>0$ and

$$\sigma^{+}(x)=\frac{1}{2\alpha}\int_{\alpha x}^{+\infty}\left((1+t)|q(t)|+\frac{2}{\alpha}|p(t)|\right)dt.$$

By the similar way, considering the solution $f_{-}(x,\lambda)$ we have for $x<0$

$$f_{-}(x,\lambda)=e^{-i\alpha\lambda x+i\omega_{-}(x)}+\int_{-\infty}^{\alpha x}A^{-}(x,t)e^{-i\lambda t}dt\ (Jm\lambda\ge 0), \quad (12)$$

where

$$\omega_{-}(x)=\frac{1}{\alpha}\int_{-\infty}^{x}p(t)dt$$

and the kernel function $A^{-}(x,t)$ satisfies the integral equation

$$A^{-}(x,t)=\frac{1}{2\alpha}\int_{-\infty}^{\frac{\alpha x+t}{2\alpha}} q(s)ds+\frac{1}{2i\alpha^2}p\left(\frac{t+\alpha x}{2\alpha}\right)e^{i\omega_{-}\left(\frac{t+\alpha x}{2}\right)}+$$

$$+\int_{-\infty}^{\frac{\alpha x+t}{2\alpha}}du\int_{-\infty}^{\frac{\alpha x-t}{2\alpha}} q(u+v)A^{+}\big(u+v,\alpha(u-v)\big)dv+$$

$$+\frac{1}{i\alpha}\int_{0}^{\frac{\alpha x-t}{2\alpha}} p\left(\frac{t+\alpha x}{2\alpha}+v\right)A^{-}\left(\frac{t+\alpha x}{2}+v,\alpha(\frac{t+\alpha x}{2}-v)\right)dv-$$

$$-\frac{1}{i\alpha}\int_{-\infty}^{\frac{\alpha x+t}{2\alpha}} p\left(u+\frac{\alpha x-t}{2\alpha}\right)A^{-}\left(u+\frac{\alpha x-t}{2\alpha},\alpha(u-\frac{\alpha x-t}{2\alpha})\right)du \qquad (13)$$

which implies

$$\int_{-\infty}^{\alpha x}|A^{-}(x,t)|\,dt\le C_1\sigma^{-}(x)e^{\sigma^{-}(x)} \qquad (14)$$

for some constant $C_1>0$ and

$$\sigma^{-}(x)=\frac{1}{2\alpha}\int_{-\infty}^{\alpha x}\left((1+t)|q(t)|+\frac{2}{\alpha}|p(t)|\right)dt.$$

Here $A^{-}(x,t)\equiv 0$ for $t>\alpha x$. Moreover, the kernel function $A^{-}(x,t)$ satisfies the condition

$$A^{-}(x,\alpha x)=\frac{1}{2\alpha}\left(\int_{-\infty}^{x}\left[q(t)+\frac{1}{\alpha^2}p^2(t)\right]dt+\frac{1}{2i\alpha}p(x)\right)e^{\omega_{-}(x)} \qquad (15)$$

As in the case of the right Jost solution we have for $x>0$

$$f_{-}(x,\lambda)=T_{+}(x)e^{i\lambda(x)}+T_{-}(x)e^{-i\lambda(x)}+\int_{-\infty}^{x}B^{-}(x,t)e^{-i\lambda t}dt\,,Im\lambda\ge 0,x>0 \qquad (16)$$

where

$$T_{\pm}(x)=\frac{1}{2}(1\mp\alpha)e^{i\omega_{-}(0)\mp i\int_0^x p(t)\,dt}$$

and

$$B^{-}(x,t)=\frac{\alpha_{-}}{2}\int_{\frac{t-x}{2\alpha}}^{0}q(s)e^{-i\omega_{-}(s)}\,ds+\frac{\alpha_{+}}{2}\int_{-\infty}^{0}q(s)e^{i\omega_{-}(s)}\,ds+\frac{1}{2}\int_{0}^{\frac{x+t}{2}}q(s)T_{+}(s)ds+$$

$$+\frac{1}{2}\int_{0}^{\frac{x-t}{2}}q(s)T_{-}(s)ds+\frac{i\alpha_{-}}{2\alpha}p\left(\frac{t-x}{2\alpha}\right)e^{i\omega_{-}\left(\frac{t-x}{2\alpha}\right)}-\frac{i}{2}p\left(\frac{t+x}{2}\right)T_{-}\left(\frac{t+x}{2}\right)+$$

$$+\frac{i}{2}p\left(\frac{x-t}{2}\right)T_{+}\left(\frac{x-t}{2}\right)+\alpha\alpha_{-}\int_{0}^{\frac{x-t}{2\alpha}}dv\int_{\frac{t-x}{2\alpha}}^{-v}q(u+v)A^{-}\big(u+v,\alpha(u-v)\big)du+$$

$$+\alpha\alpha_{+}\int_{0}^{\frac{x-t}{2\alpha}}dv\int_{-\infty}^{-v}q(u+v)A^{-}\big(u+v,\alpha(u-v)\big)du+$$

$$+\int_{0}^{\frac{x-t}{2}}dv\int_{-v}^{\frac{x+t}{2}}q(u+v)B^{-}(u+v,u-v)du+$$

$$+i\alpha_{-}\int_{0}^{\frac{x-t}{2\alpha}}p(\frac{t-x}{2\alpha}+v)A^{-}(\frac{t-x}{2\alpha}+v,\alpha\left(\frac{t-x}{2\alpha}-v\right)dv+$$

$$+i\alpha_+ \int_{-\infty}^{\frac{x-t}{2\alpha}} p(u-\frac{t-x}{2\alpha})A^-(u-\frac{t-x}{2\alpha},\alpha\left(\frac{t-x}{2\alpha}+u\right)du +$$

$$+i \int_{-\frac{x-t}{2}}^{\frac{x+t}{2}} p(u+\frac{x-t}{2})B^-(u+\frac{x-t}{2},\alpha\left(u-\frac{x-t}{2}\right)dv -$$

$$-i \int_{0}^{\frac{x-t}{2}} p\left(\frac{x+t}{2}+v\right)B^-\left(\frac{x+t}{2}+v,\frac{x+t}{2}-v\right)dv, -x<t\le x, x>0 \qquad (17)$$

$$B^-(x,t) = \frac{\alpha_+}{2}\int_{-\infty}^{\frac{t+x}{2\alpha}} q(s)e^{-i\omega_-(s)}\,ds - \frac{\alpha_-}{2}\int_{-\infty}^{\frac{t-x}{2\alpha}} q(s)e^{i\omega_-(s)}\,ds +$$

$$\frac{i\alpha_-}{2\alpha}p\left(\frac{t-x}{2\alpha}\right)e^{i\omega_-\left(\frac{t-x}{2\alpha}\right)} - \frac{i\alpha_+}{2\alpha}p\left(\frac{t+x}{2\alpha}\right)e^{i\omega_-\left(\frac{t-x}{2\alpha}\right)} +$$

$$+\alpha_-\alpha \int_{-\frac{x+t}{2\alpha}}^{\frac{x-t}{2\alpha}} dv \int_{-\frac{x-t}{2\alpha}}^{-v} q(u+v)A^-(u+v,u-v)du -$$

$$-\alpha_-\alpha \int_{0}^{-\frac{x+t}{2\alpha}} dv \int_{-\infty}^{\frac{t-x}{2\alpha}} q(u+v)A^-\big(u+v,\alpha(u-v)\big)du +$$

$$+\alpha\alpha_+ \int_{0}^{-\frac{x+t}{2\alpha}} dv \int_{-\infty}^{\frac{t-x}{2\alpha}} q(u+v)A^-\big(u+v,\alpha(u-v)\big)du +$$

$$+\alpha\alpha_+ \int_{-\frac{x+t}{2\alpha}}^{\frac{x-t}{2\alpha}} dv \int_{-\infty}^{-v} q(u+v)A^-\big(u+v,\alpha(u-v)\big)du +$$

$$+ \int_{-\frac{x+t}{2\alpha}}^{\frac{x-t}{2}} dv \int_{-v}^{\frac{x+t}{2}} q(u+v)B^-(u+v,u-v)du -$$

$$-i\alpha_- \int_{-\infty}^{\frac{t+x}{2\alpha}} p\left(u-\frac{t+x}{2\alpha}\right)A^-\left(u-\frac{t+x}{2\alpha},u+\frac{t+x}{2\alpha}\right)du +$$

$$+i\alpha_- \int_0^{\frac{x-t}{2\alpha}} p\left(\frac{t-x}{2\alpha}+v\right) A^-\left(\frac{t-x}{2\alpha}+v, \alpha\left(\frac{t-x}{2}-v\right)\right) dv +$$

$$-i\alpha_+ \int_{-\infty}^{\frac{t-x}{2\alpha}} p\left(u+\frac{x-t}{2\alpha}\right) A^-\left(u+\frac{x-t}{2\alpha}, \alpha\left(u-\frac{x-t}{2\alpha}\right)\right) du -$$

$$-i\alpha_+ \int_0^{-\frac{x+t}{2\alpha}} p\left(\frac{x+t}{2\alpha}+v\right) A^-\left(\frac{x+t}{2\alpha}, \alpha\left(\frac{x+t}{2\alpha}-v\right)\right) du +$$

$$+i \int_{\frac{t-x}{2}}^{\frac{t+x}{2}} p\left(u-\frac{t-x}{2}\right) B^-\left(u-\frac{t-x}{2}, u+\frac{t-x}{2}\right) du - \tag{18}$$

$$-i \int_0^{\frac{t-x}{2}} p\left(\frac{x+t}{2}-v\right) B^-\left(\frac{x+t}{2}-v, \frac{x+t}{2}+v\right) dv, \quad t<-x<0$$

and $B^-(x,t) \equiv 0$ for $t > x$. Estimating (10), (11) and (17), (18) we can easily obtain that

$$\int_{-\infty}^{0} |B^-(x,t)| \leq C_1 \sigma^-(x) e^{\sigma^-(x)} \tag{19}$$

for some $C_1 > 0$ where

$$\sigma^-(x) = \left(\int_{-\infty}^{x} ((1+t)\,|q(t)| + 2|p(t)|)\right) dt.$$

Hence by setting $K^{\pm}(x,t) = \begin{cases} A^{\pm}(x,t), & \pm x \geq 0 \\ B^{\pm}(x,t), & \pm x < 0 \end{cases}$ and combining all our results we have the following theorem.

Theorem 1. *For all $Im\,\lambda \geq 0$ the solutions $f_+(x,\lambda)$ and $f_-(x,\lambda)$ can be represented as*

$$f_+(x,\lambda) = R_+(x) e^{i\lambda\mu(x)} + R_-(x) e^{-i\lambda\mu(x)} + \int_{\mu(x)}^{+\infty} K^+(x,t) e^{i\lambda t} dt$$

and

$$f_-(x,\lambda) = T_+(x) e^{i\lambda\mu(x)} + T_-(x) e^{-i\lambda\mu(x)} + \int_{-\infty}^{\mu(x)} K^-(x,t) e^{-i\lambda t} dt$$

respectively, where $\mu(x) = x\sqrt{\rho(x)}$

$$R_+(x) = \frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right) e^{i\int_x^{+\infty} \frac{\rho(t)}{\sqrt{\rho(t)}} dt}$$

$$R_-(x) = \frac{1}{2}(1 - \frac{1}{\sqrt{\rho(x)}})e^{i\int_x^{+\infty}\frac{\rho(t)sgnt}{\sqrt{\rho(t)}}dt}$$

$$T_+(x) = \frac{1}{2}(1 - \frac{\alpha}{\sqrt{\rho(x)}})e^{-i\int_{-\infty}^{x}\frac{\rho(t)sgnt}{\sqrt{\rho(t)}}dt}$$

$$T_-(x) = \frac{1}{2}(1 + \frac{\alpha}{\sqrt{\rho(x)}})e^{i\int_{-\infty}^{x}\frac{\rho(t)}{\sqrt{\rho(t)}}dt}$$

From the estimations (6), (111), (14) and (19) we have

$$\int_{\mu(x)}^{+\infty} |K^+(x,t)|dt \le C\sigma^+(\mu(x))e^{\sigma^+(\mu(x))} \tag{20}$$

and

$$\int_{-\infty}^{\mu(x)} |K^+(x,t)|dt \le C\sigma^-(\mu(x))e^{\sigma^-(\mu(x))} \tag{21}$$

for some $C > 0$. Now let

$$G^+(x,t) = -\int_t^{+\infty} K^+(x,s)ds,$$

$$G^-(x,t) = -\int_{-\infty}^{t} K^+(x,s)ds.$$

Then we have

$$f_+(x,\lambda) = R_+(x)e^{i\lambda\mu(x)} + R_-(x)e^{-i\lambda\mu(x)} + \int_{\mu(x)}^{+\infty} G_t{}^+(x,t)e^{i\lambda t}dt$$

$$= R_+(x)e^{i\lambda\mu(x)} + R_-(x)e^{-i\lambda\mu(x)} - G^+\big(x,\mu(x)\big)e^{i\lambda\mu(x)} -$$

$$-\big(G^+(x,-\mu(x)+0) - G^+(x,-\mu(x)-0)\big)e^{-i\lambda\mu(x)} - i\lambda\int_{\mu(x)}^{+\infty} G^+(x,t)e^{i\lambda t}dt$$

and

$$f_-(x,\lambda) = T_+(x)e^{i\lambda\mu(x)} + T_-(x)e^{-i\lambda\mu(x)} + \int_{-\infty}^{\mu(x)} G_t{}^-(x,t)e^{-i\lambda t}dt$$

$$= T_+(x)e^{i\lambda\mu(x)} + T_-(x)e^{-i\lambda\mu(x)} - G^+\big(x,\mu(x)\big)e^{-i\lambda\mu(x)} -$$

$$-\big(G^-(x,-\mu(x)+0) - G^-(x,-\mu(x)-0)\big)e^{i\lambda\mu(x)} - i\lambda\int_{-\infty}^{\mu(x)} G^-(x,t)e^{-i\lambda t}dt$$

Therefore, by setting $\lambda = 0$ in the last formulas we have

$$f_+(x,\lambda) = L^+(x)e^{i\lambda\mu(x)} + L_-(x)e^{-i\lambda\mu(x)} - i\lambda \int_{\mu(x)}^{+\infty} G^+(x,t)e^{i\lambda t}dt \tag{22}$$

and

$$f_-(x,\lambda) = P_+(x)e^{i\lambda\mu(x)} + P_-(x)e^{-i\lambda\mu(x)} - i\lambda \int_{-\infty}^{\mu(x)} G^-(x,t)e^{-i\lambda t}dt \tag{23}$$

where

$$L^+(x) = R_+(x) - G^+\big(x,\mu(x)\big),$$
$$L_-(x) = R_-(x) - \big(G^+(x,-\mu(x)+0) - G^+(x,-\mu(x)-0)\big)$$

and

$$P^+(x) = T_-(x) + G^+\big(x,\mu(x)\big),$$
$$P_-(x) = T_-(x) - \big(G^-(x,-\mu(x)+0) - G^-(x,-\mu(x)-0)\big).$$

Theorem 2. *If $(1+|x|)q(x), p(x) \in L^1(I)$ then the Jost solutions $f_+(x,\lambda)$ and $f_-(x,\lambda)$ are expressed as (22) and (23) respectively, where the kernels $G^{\pm}(x,t)$ are bounded functions on I Additionally, if $p(x) \in BC(I)$ then $G_t{}^{\pm}(x,t)$ are bounded for $\pm\mu(x) \leq \pm t$ as well as $G_t{}^{+}(x,t) \in L_1(\mu(x),+\infty)$ and $G_t{}^{-}(x,t) \in L_1\big(-\infty,\mu(x)\big)$. Moreover the following relations are satisfied:*

$$G_t{}^{+}\big(x,\mu(x)\big) = R_+(x)\left\{\frac{1}{2}\int_x^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)ds + \right.$$
$$\left. +\frac{i}{2}\int_x^{+\infty}\left[\left(\frac{1}{\sqrt{\rho(x)}}\right)^2 + \left(1 - \frac{\sqrt{\rho(s)}}{\sqrt{\rho(x)}}\right)^2\right]p'(s)ds\right\}, \tag{24}$$

$$G_t{}^{+}(x,\mu(x)+0) - G_t{}^{+}(x,-\mu(x)-0) = R_-(x)\left\{\frac{1}{2}\int_x^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)sgns ds + \right.$$
$$+\frac{i}{2}\int_x^{+\infty}\left[\left(\frac{1}{\sqrt{\rho(x)}}\right)^2\right. \tag{25}$$
$$\left.\left. + \left(1 + \frac{sgns\sqrt{\rho(s)}}{\sqrt{\rho(x)}}\right)^2\right]p'(s)ds\right\},$$

$$G_t{}^{-}\big(x,\mu(x)\big) = T_-(x)\left\{\frac{1}{2}\int_{-\infty}^{x}\left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)ds - \right.$$
$$\left. -\frac{i}{2}\int_{-\infty}^{x}\left[\left(\frac{1}{\sqrt{\rho(x)}}\right)^2 + \left(1 - \frac{\sqrt{\rho(x)}}{\sqrt{\rho(s)}}\right)^2\right]p'(s)ds\right\}, \tag{26}$$

$$G_t{}^{-}(x,-\mu(x)+0) - G_t{}^{-}(x,-\mu(x)-0) =$$
$$= T_+(x)\left\{\frac{1}{2}\int_{-\infty}^{x}\left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)sgns ds + \right. \tag{27}$$

$$+\frac{i}{2}\int_{-\infty}^{x}\left[1+\left(1-\frac{sgns\sqrt{\rho(x)}}{\sqrt{\rho(s)}}\right)^{2}\right]p'(s)ds\Bigg\}.$$

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